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# Lamb Shifts for Hydrogen, Using the Sachs Elementary Interaction Theory

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*Abstract*

The Lamb shifts of the $S_{1/2}$ and $P_{1/2}$ states of hydrogen atoms are calculated using the Sachs elementary interaction theory. Both $S_{1/2}$ and $P_{1/2}$ levels are shifted with respect to the Dirac levels by energies of the same order of magnitude. Agreement with experiment is obtained for the $^2S_{1/2} - {}^2P_{1/2}$ and $^3S_{1/2} - {}^3P_{1/2}$ Lamb shifts, but the predicted $^1S_{1/2} - {}^2P_{1/2}$ Lamb shift term of 12,164 MHz is in disagreement with the experimental value of $7860 \pm 1140$ MHz.

## 1. *Introduction*

According to the Dirac theory for hydrogenic atoms, the $^nS_{1/2}$ and $^nP_{1/2}$ energy levels are degenerate. Lamb and Retherford (1947) demonstrated by microwave methods that the $^2S_{1/2}$ level was displaced upwards relative to the $^2P_{1/2}$ level by a small amount of the order of 1000 MHz. Subsequent experiments (Triebwasser *et al.*, 1953) have yielded extremely accurate values for the Lamb shifts.

In quantum electrodynamics, the Lamb shift is caused by the interaction of the electron with the zero-point vibrations of the radiation field [Bethe 1947; Weisskopf, 1949]. This interaction is different for free and bound electrons, and is greatest for $S$ states and very small for $P, D, \ldots$ states.

The quantum electrodynamic formula for the $S$ state shift is (Herzberg, 1956)

$$\Delta E(n, 0) = \frac{8Z^4\alpha^3}{3\pi n^3} R_\infty \left(1 - \frac{3m}{M}\right)\left[\ln \frac{mc^2}{2k_0(n, 0)} + \frac{19}{30} + 3\pi Z\alpha\left(\frac{427}{384} - \frac{1}{2}\ln 2\right)\right] \tag{1.1}$$

where $n$ = principal quantum number, $Z$ = nuclear charge, $\alpha$ = fine structure constant, $m$ = electronic mass, $M$ = nuclear mass, $R_\infty$ = Rydberg constant, for infinite mass, and $k_0(n, 0)$ an average excitation energy, having the value $16.646\, R_\infty hc$ for the $^2S_{1/2}$ state, and $19.76967\, R_\infty hc$ for the $^1S_{1/2}$ state.

For $P$ states, the shift is (Bethe *et al.*, 1949)

$$\Delta E(n, l) = \frac{8Z^4\alpha^3}{3\pi n^3} R_\infty \left(1 - \frac{3m}{M}\right)\left[\ln \frac{R_\infty hc}{k_0(n, l)} + \frac{3}{8}\frac{c_{lj}}{2l+1}\right]$$

where

$$c_{lj} = \frac{1}{l+1} \text{ for } j = l + \frac{1}{2}$$
$$= -\frac{1}{l} \text{ for } j = l - \frac{1}{2} \tag{1.2}$$

and $k_0(2, 1) = .9704\, hc$.

Equations (1.1) and (1.2) result from a calculation where there are divergent integrals, which are removed by appealing to renormalization schemes and external physical constraints such as gauge invariance.

Sachs (1971a, b; 1972a, b) has formulated a theory of elementary matter in which the Lamb shift is the result of a small finite modification of the Coulomb potential between electron and nucleus. His elementary interaction theory has Lorentz invariance built in from the initial axioms, and there are no divergent integrals in the calculations. A new fundamental constant of length, $g_M$, results from including explicitly the matter field variables included in the electromagnetic quantities $\rho$ and $j$. In the explicit calculation of the Lamb shift in hydrogen (Sachs, 1972b), the ratio $\Delta(^3S_{1/2} - {}^3P_{1/2})/\Delta(^2S_{1/2} - {}^2P_{1/2})$ is found to be in agreement with experiment, independent of the value of $g_m$. The experimental result for $\Delta(^2S_{1/2} - {}^2P_{1/2})$ is then used to fix the value of $g_M$. The calculation has only been performed for those Lamb shifts between $S_{1/2}$ and $P_{1/2}$ states corresponding to the same principal quantum number, and both $\Delta(^2S_{1/2} - {}^2P_{1/2})$ and $\Delta(^3S_{1/2} - {}^3P_{1/2})$ agree with experiment.

In this paper, we use the elementary interaction theory to calculate the shifts of $S_{1/2}$ and $P_{1/2}$ levels from the corresponding Dirac levels. We find that the two shifts are of the same order of magnitude, unlike in quantum electrodynamics where the shift of the $S_{1/2}$ level is much greater than that of the $P_{1/2}$ level. From these shifts we calculate the Lamb shift term $\Delta(^nS_{1/2} - {}^mP_{1/2})$ for a general $n, m (n \neq m)$. The particular case $\Delta(^1S_{1/2} - {}^2P_{1/2})$ is in conflict with the experimental result of Herzberg (1956).

## 2. *Calculation of the Lamb Shift*

Using the notation of Sachs (1972b) we consider the solution of the coupled equations

$$(\gamma_\mu \partial^\mu - I(p) + \lambda)\psi^{(e)} = 0 \tag{2.1}$$

$$(\gamma_\mu \partial^\mu + I(e) + \Lambda)\psi^{(p)} = 0 \tag{2.2}$$

where $\gamma_\mu$ are the Dirac matrices, $\lambda$, $\Lambda$ are the reciprocal Compton wavelengths for electron and proton respectively, and $I$ describes the interaction potential between proton and electron.

Equations (2.1) and (2.2) are nonlinear equations; however, assuming both a point proton and a negligible momentum transfer from electron to proton, the coupled equations can be reduced to the following linearized form (Sachs, 1972b)

$$(H_0 + \hat{V})\psi^{(e)} = -\frac{iE}{\eta}\psi^{(e)} \tag{2.3}$$

where $H_0$ is the unperturbed Dirac Hamiltonian for the hydrogen atom, and

$$\hat{V} = \frac{-i\kappa}{\rho^2}\frac{(\rho \times \alpha)_3}{\rho}$$

where

$$\kappa = \frac{16\pi}{[(s+n-1)]^2 + \alpha^2}\left(\frac{g_M}{\lambda_c}\right)\alpha^2$$

is a measure of the strength of the interaction $I(p)$. $\lambda_c$ is the reduced Compton wavelength, $\hbar/mc$, for the electron, and $g_M$ the fundamental length constant.

The potential $\hat{V}$ lacks reflection symmetry in both space and time by virtue of the outer product, and so will lift the accidental degeneracy in the eigenstates of $H_0$. In his solution of (2.3), Sachs solves

$$(H_0{}^1 + V^1)\psi^{(e)} = -\frac{iE}{\eta}\psi^{(e)}$$

where

$$H_0{}^1 = H_0 - \frac{\kappa\alpha_\rho}{\rho^2}$$

and

$$V^1 = \hat{V} + \frac{\kappa\alpha_\rho}{\rho^2} \tag{2.4}$$

The shifts of the $S_{1/2}$ and $P_{1/2}$ levels from the Dirac levels will be given by $\langle S_{1/2}|iV^1|S_{1/2}\rangle$ and $\langle P_{1/2}|iV^1|P_{1/2}\rangle$, where $|S_{1/2}\rangle$ and $|P_{1/2}\rangle$ are given by $|S_{D,1/2}\rangle \exp(-\kappa/\rho)$ and $|P_{D,1/2}\rangle \exp(-\kappa/\rho)$. Substituting the explicit expressions for the Dirac eigenfunctions (Bethe and Salpeter, 1957)

$$|^nS_{1/2}\rangle = \begin{pmatrix} F_-(n-1) \\ 0 \\ -i\cos\theta\ G_-(n-1) \\ -i\sin\theta\exp(i\phi)G_-(n-1) \end{pmatrix}\exp(-\kappa/\rho) \tag{2.5}$$

$$|{}^nP_{1/2}\rangle = \begin{pmatrix} \cos\theta F_+(n-1) \\ \sin\theta \exp(i\phi) F_+(n-1) \\ -iG_+(n-1) \\ 0 \end{pmatrix} \exp(-\kappa/\rho) \tag{2.6}$$

where ${}_\eta F_\pm/\rho$ and ${}_\eta G_\pm/\rho$ are respectively the large and small parts of the solutions of the radial Dirac equation, and $n$ is the principal quantum number, the matrix elements are readily evaluated. We obtain

$$\langle |{}^nS_{1/2}|i\hat{V}^1|{}^nS_{1/2}\rangle = \frac{64\pi}{3[(s+n-1)^2+\alpha^2]}\left(\frac{g_M}{\lambda_c}\right)\alpha^4(mc^2)\,|I_{-,n-1}| \tag{2.7}$$

and

$$\langle {}^mP_{1/2}|i\hat{V}^1|{}^mP_{1/2}\rangle = \frac{64\pi}{3[(s+m-1)^2+\alpha^2]}\left(\frac{g_M}{\lambda_c}\right)\alpha^4(mc^2)I_{+,m-1} \tag{2.8}$$

where

$$I_{\pm,n-1} = \frac{1}{\eta\alpha}\int_0^\infty \frac{F_\pm(n-1)G_\pm(n-1)}{\rho^2}\exp(-2\kappa/\rho)\,d\rho \tag{2.9}$$

The integrals $I_\pm$ are evaluated in terms of modified Bessel functions (Sachs, 1972b) and numerical values given in Table 1. Combining (2.7) and (2.8) we obtain the values for the Lamb shift term for the transition ${}^nS_{1/2} \to {}^mP_{1/2}$ as

$$\Delta({}^nS_{1/2} - {}^mP_{1/2}) = \frac{64\pi}{3}\left(\frac{g_M}{\lambda_c}\right)\alpha^4(mc^2)\left[\frac{|I_{-,n-1}|}{(s+n-1)^2} - \frac{I_{+,m-1}}{(s+m-1)^2}\right] \tag{2.10}$$

The constant $g_M$ in (2.10) is obtained from the experimental result $\Delta({}^2S_{1/2} \to {}^2P_{1/2}) = 1057.77$ MHz (Triebwasser *et al.*, 1953), yielding $g_M = 2.087 \times 10^{-14}$ cm.

The result (2.10) can be considered as an upper limit to any particular Lamb shift. If the point proton approximation were not applicable the proton field is spread over a finite volume of space and the matrix elements would be smaller. This reduction would apply particularly to the matrix element $\langle {}^1S_{1/2}|i\hat{V}^1|{}^1S_{1/2}\rangle$.

TABLE 1. Integrals of the form $I_\pm n-1$

| $n$ | $I_{+,n-1}$ | $I_{-,n-1}$ |
|---|---|---|
| 1 | 0 | −1 |
| 2 | 1/6 | −2 |
| 3 | 1/9 | −3 |
| 4 | 1/12 | −4 |

### 3. *Discussion*

From equations (1.1) and (1.2), the shifts of the $^2S_{1/2}$ and $^2P_{1/2}$ states from the Dirac level are 1044.83 MHz and −12.85 MHz, ignoring all corrections of $\alpha^5$ and higher. The $^1S_{1/2}$ Lamb shift is 8172 MHz, and so the Lamb shift term in $\Delta(^1S_{1/2} - {}^2P_{1/2})$ is 8185 MHz to fourth order in $\alpha$, according to quantum electrodynamics.

With the elementary interaction theory the $^1S_{1/2} - {}^2P_{1/2}$ Lamb shift is obtained by putting $n = 0$, $m = 1$ into (2.10), yielding

$$\Delta(^1S_{1/2} - {}^2P_{1/2}) = \frac{64}{3}\left(\frac{g_M}{\lambda_c}\right)\alpha^4(mc^2)\left[\frac{|I_{-,0}|}{1} - \frac{I_{+,1}}{4}\right] + 0(\alpha^6) \quad (3.1)$$

Now $I_{+,1} = \frac{1}{6}$, $I_{-,0} = -1$ from Table 1, so

$$\Delta(^1S_{1/2} - {}^2P_{1/2}) = \frac{64\pi}{3}\left(\frac{g_M}{\lambda_c}\right)\alpha^4(mc^2)\left[\frac{23}{24}\right] \quad (3.2)$$

Now $g_M$ is eliminated from (3.2) by setting $\Delta(^2S_{1/2} - {}^2P_{1/2})$ equal to the experimental value of 1057.77 MHz. The final result is 12,164 MHz for $\Delta(^1S_{1/2} - {}^2P_{1/2})$.

Direct comparison with experiment for hydrogen is impossible. Herzberg (1956) measured the Lamb shift $\Delta(^2P - {}^1S_{1/2})$ in deuterium to be 7860 MHz, with upper and lower values of $\Delta$ being 9010 MHz and 4830 MHz. The very low value of the lower bound assumes that the weak-absorption assumption made by Herzberg was invalid. Since, according to quantum electrodynamics, the $P$ level is only shifted by a small amount, the failure of Herzberg to resolve the $^2P_{3/2}$ and $^2P_{1/2}$ states only meant that separate values for $\Delta(^2P_{3/2} - {}^1S_{1/2})$ and $\Delta(^2P_{1/2} - {}^1S_{1/2})$ could not be obtained, but that $(^2P - {}^1S_{1/2})$ was a good approximation for either shift.

If the point nucleus approximation used in the derivation of (2.10) were valid, the elementary interaction result for deuterium would be altered only in the factor $\lambda_c$ appearing in (2.10), yielding a result of 12,170 MHz for the shift $\Delta(^2P_{1/2} - {}^1S_{1/2})$ in deuterium.

The considerable discrepancy between theory and experiment could be traced to the failure of the point-nucleus linear approximation used in the derivation of (2.10) for the case of the $^1S_{1/2}$ shift. In addition, the elementary interaction theory predicts that the $S_{1/2}$ states are shifted by an energy given by (2.7) and that the $P_{1/2}$ states have an energy shift given by (2.8). For $n \neq 1$, where the point nucleus approximation should be valid, these shifts are of the same order of magnitude e.g., for $n = 2$, $\langle ^2S_{1/2}|i\hat{V}^1|^2S_{1/2}\rangle = 1586.66$ MHz and $\langle ^2P_{1/2}|i\hat{V}^1|^2P_{1/2}\rangle = 528.89$ MHz unlike the quantum electrodynamic prediction of a very small $P_{1/2}$ shift. Thus discrepancies with quantum electrodynamics are not confined to $n = 1$ states.

To make the predictions of the elementary interaction theory consistent with experiment for both the $^1S_{1/2} - {}^2P_{1/2}$ and the $^2S_{1/2} - {}^2P_{1/2}$ Lamb shifts, the matrix element $\langle P_{1/2}|i\hat{V}^1|P_{1/2}\rangle$ must be set equal to zero (at least to fourth

order in $\alpha$) either by modification of the perturbing Hamiltonian $i\hat{V}$, or by appeal to some symmetry property of the atomic system. If this should be possible, the value of the fundamental constant $g_M$ would be determined by the equating of $\langle {}^2S_{1/2}|i\hat{V}^1|{}^2S_{1/2}\rangle$ and the experimental value of the ${}^2S_{1/2} - {}^2P_{1/2}$ Lamb shift. From this we would find $g_M' = 1.391 \times 10^{-14}$ cm and $({}^1S_{1/2} - {}^2P_{1/2}) = 8462$ MHz.

A determination of $g_M$, independent of the Lamb shift, can be obtained from the observed minimum in the elastic scattering cross-section of the $e - He^4$ system. (Sachs, 1968, Frosch *et al.*, 1967). Fitting the zero of the calculated equivalent form factor $\epsilon(g)$ to the observed minimum cross-section at $q = 3.2 \times 10^{-13}$ cm$^{-1}$ yields a value of $g_M$ of $3.19 \times 10^{-14}$ cm from scattering data. Thus we cannot use the present scattering experiments to resolve the ${}^1S_{1/2}$ Lamb shift discrepancy by a mere modification of the magnitude of the fundamental constant $g_M$.

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